- Vector Space Axioms: Let V be a set on which the operations of addition $\mathbf{x}+\mathbf{y}$ and scalar multiplication $\alpha\mathbf{x}$ are defined for all $\mathbf{x},\mathbf{y}\in V$ and scalar $\alpha\in\mathbb{R}$. The set V together with the operations of addition and scalar multiplication is said to form a vector space if the following axioms are satisfied:

  C1. If $\mathbf{x}\in V$ and α is a scalar, then $\alpha\mathbf{x}\in V$

  C2. If $\mathbf{x},\mathbf{y}\in V$, then $\mathbf{x}+\mathbf{y}\in V$

  A1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$

  A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$

  A3. There exists an element $\mathbf{0}$ in V such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for each $\mathbf{x}\in V$

  A4. For each $\mathbf{x}\in V$ there exists an element $-\mathbf{x}\in V$ such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.

  A5. $\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}$

  A6. $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$

  A7. $(\alpha\beta)\mathbf{x}=\alpha(\beta\mathbf{x})$

  A8. $1\mathbf{x}=\mathbf{x}$

- Subspace: If S is a nonempty subset of a vector space V, and S satisfies the conditions

  (i) $\alpha\mathbf{x}\in S$ whenever $\mathbf{x}\in S$ for any scalar α

  (ii) $\mathbf{x}+\mathbf{y}\in S$ whenever $\mathbf{x}\in S$ and $\mathbf{y}\in S$

  then S is said to be a subspace of V.

- Let $v_1,v_2,\cdots,v_n$ be vectors in a vector space V. The set $\{\alpha_1v_1+\alpha_2v_2+\cdots\alpha_nv_n\,|\,\alpha_1,\cdots,\alpha_n\in\mathbb{R}\}$ is called the span of $v_1,v_2,\cdots,v_n$ and is denoted by $\mathrm{Span}(v_1,v_2,\cdots,v_n)$.

- The vectors $v_1,v_2,\cdots,v_n$ in a vector space V are said to be linearly independent if $c_1v_1+c_2v_2+\cdots c_nv_n=\mathbf{0}$ implies that all the scalars $c_1=c_2=\cdots=c_n=0$.

  Theorem 0.1. *Let $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n$ be n vectors in $\mathbb{R}^n$ and let $X=(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n)$. The vectors $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n$ will be linearly independent and span $\mathbb{R}^n$ if and only if X is nonsingular.*

- If vectors $v_1,v_2,\cdots,v_n$ are linearly independent and span V, then $v_1,v_2,\cdots,v_n$ form a basis for V and V has dimension n.

  Theorem 0.2. *If vectors $v_1,v_2,\cdots,v_n$ form a basis for V, then any collection of (strictly) more than n vectors in V, is linearly dependent.*

- Let $\{\mathbf{e}_1,\mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and $\{\mathbf{u}_1,\mathbf{u}_2\}$ be another ordered basis. $U=(\mathbf{u}_1,\mathbf{u}_2)$ is called the transition matrix from $\{\mathbf{u}_1,\mathbf{u}_2\}$ to $\{\mathbf{e}_1,\mathbf{e}_2\}$. If $\{\mathbf{v}_1,\mathbf{v}_2\}$ is also an ordered basis and V is the transition matrix from $\{\mathbf{v}_1,\mathbf{v}_2\}$ to $\{\mathbf{e}_1,\mathbf{e}_2\}$. Then the transition matrix from $\{\mathbf{v}_1,\mathbf{v}_2\}$ to $\{\mathbf{u}_1,\mathbf{u}_2\}$ is $U^{-1}V$

- The rank of a matrix A, denoted $\mathrm{rank}(A)$, is the number of non-zero rows in the reduced echelon form of A. The dimension of the null space of a matrix is called the nullity of the matrix.

  Theorem 0.3. *If A is an $m\times n$ matrix, then the rank of A plus the nullity of A equals n.*

- For an $n\times n$ matrix $A=(a_{ij})$, $p(\lambda)=\det(A-\lambda I)$ is called the characteristic polynomial of A. $\mathrm{tr}(A)=\sum_{i=1}^n a_{ii}$ is called the trace of A.

  Theorem 0.4. *If $\lambda_1,\lambda_2,\cdots,\lambda_n$ are the eigenvalues of A, then*

$$\lambda_1\cdot\lambda_2\cdots\lambda_n=p(0)=\det(A) \tag{0.1}$$

$$\lambda_1+\lambda_2+\cdots+\lambda_n=\mathrm{tr}(A)=\sum_{i=1}^n a_{ii} \tag{0.2}$$

  Theorem 0.5. *Let A and B be two $n\times n$ matrices. If there is a nonsingular matrix S such that $B=S^{-1}AS$, then A and B have the same characteristic polynomial and the the same eigenvalues.*

- An $n\times n$ matrix A is said to be diagonalizable if there exists a nonsingular matrix X and a diagonal matrix D such that $X^{-1}AX=D$. We say that X diagonalizes A.

  Theorem 0.6. *An $n\times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.*

- A mapping L from a vector space V into a vector space W is said to be a linear transformation if for all $v_1,v_2\in V$ and all scalars α

  (i) $L(v_1+v_2)=L(v_1)+L(v_2)$

  (ii) $L(\alpha v_1)=\alpha L(v_1)$

- Let $L:V\to W$ be a linear transformation. Let $\mathbf{0}_V$ and $\mathbf{0}_W$ be the zero vectors in V and W, respectively. The kernel of L, denoted $\ker(L)$, is defined by

$$ker(L)=\{v\in V|L(v)=\mathbf{0}_W\}$$

  Let S be a subspace of V. The image of S, denoted $L(S)$, is defined by

$$L(S)=\{L(v)|v\in S\}$$

  The image of the entire vector space, $L(V)$, is called the range of L.

- Let $L:\mathbb{R}^n\to\mathbb{R}^m$ be a linear transformation. An $m\times n$ matrix A is called the (standard) matrix representation of A if

$$L(\mathbf{x})=A\mathbf{x},\quad \mathbf{x}\in\mathbb{R}^n$$

  Theorem 0.7. *For any linear transformation $L:\mathbb{R}^n\to\mathbb{R}^m$, L has an $m\times n$ matrix representation A. Moreover,*

$$A=(L(\mathbf{e}_1),L(\mathbf{e}_2),\cdots,L(\mathbf{e}_n))$$

  where $\mathbf{e_1},\cdots,\mathbf{e_n}$ is the standard basis of $\mathbb{R}^n$.